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Recursively Indexed Differential Pulse Code Modulation*

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Abstract

We study the performance of a DPCM system with a recursively indexed quantizer (RIQ) under various conditions, with first order Gauss-Markov and Laplace-Markov sources as inputs. We show that when the predictor is matched to the input, the proposed system performs at or close to the optimum entropy constrained DPCM system. We also show that the if we are willing to accept a 5% increase in the rate, the system is very forgiving of predictor mismatch.

1 Introduction

Differential pulse code modulation (DPCM) is often used to efficiently convert an analog source such as speech, music or images into a digital form for communication or storage. Its efficiency is due to the exploitation of the memory in a source by the use of a predictor, which estimates the present source sample to be encoded, based on the quantized previous source samples. The performance of DPCM depends on two factors

1. How well the predictor exploits the source memory, i.e. , how closely it can estimate the actual source samples.
2. How well the quantizer is matched to the prediction error (the quantizer input).

In order to maximize the goodness of prediction, the predictor is usually chosen based on the statistical properties of a given source. However, many physical sources, such as those listed above, exhibit statistically varying local properties which are usually quite distinct from their global ones. If this is the case and DPCM happens to operate on a segment whose statistics differ from the global ones, it operates in a mismatched state, which results in additional degradation in the reproduction [1].

Various schemes have been devised to handle this mismatch between the source and the predictor. These involve some form of adaptation of the quantizer and/or the predictor, by which DPCM quickly follows up the changing statistics of the source and prevents overloading of the system. However, this quick response of DPCM is not without its cost: the adaptation requires more implementation and operational complexity.

Matching the quantizer to the statistics of the prediction error is even more difficult, as the quantizer structure itself effects the statistical properties of the prediction error. One way to obtain the statistics of the prediction error process is through the use of an orthonormal expansion [2, 3, 4, 5]. This has been used to optimize the quantizer through an iterative procedure [4, 6, 7, 8]. However, under operational circumstances this might not be a viable option.

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In this paper we study the performance of a DPCM system operating on a Gauss-Markov source and a Laplace-Markov source. The DPCM system considered here, called a recursively indexed DPCM, consists of a uniform quantizer with infinitely many output levels, a recursively indexed binary encoder, and a first-order linear predictor. The quantizer is designed simply by specifying its step-size, and the predictor is non-adaptive. The Gauss-Markov and Laplace-Markov sources are chosen because of their use in the modeling of physical sources. The goal is to observe the rate distortion performance of this system, as well as the performance degradation when the predictor and the source are mismatched. We compare the rate distortion performance of the proposed system to the optimum results in the literature [6], where the quantizer was optimized using the iterative procedure mentioned above.

The simulation results show that the rate distortion performance of the proposed system achieves or comes very close to the optimum performance at all rates studied. In the case of mismatch the simulation results show that only a 5 percent increase in the rate allows a rather wide range of the predictor mismatch to first-order Gauss-Markov and Laplace-Markov sources. They agree with a result in [9] reported for a 2-level optimized DPCM that the predictor coefficient does not significantly effect the optimality of DPCM. But unlike in [9] it is observed that a lower predictor coefficient than the source correlation coefficient is better for the low rate case.

In Section 2 a recursively indexed binary encoder is discussed. In Section 3 DPCM is briefly reviewed and the DPCM mismatch problem is posed. In Section 4 the performance of a recursively indexed DPCM is considered and numerical results are presented. Conclusions follow in section 5.

2 A Recursively Indexed Binary Encoder

The DPCM system discussed here uses a quantizer with infinitely many output levels. This requires binary encoding of a countably infinite alphabet, which poses obvious problems in design and operation. An obvious and reasonable approach is first to represent the input alphabet using only finite many symbols and then to encode these symbols either using a fixed-to-fixed or fixed-to-variable length encoding. A recursively indexed binary encoder is used for just this purpose.

The recursively indexed binary encoder considered in this paper is a two stage binary encoder: recursive indexing followed by an optimum (the minimum average codeword length) symbol-to-variable length binary encoder for the output of recursive indexing.

Recursive indexing is a mapping of a countable set to a collection of sequences of symbols from another set of finite size [10, 11]. Given a countable set $A = \{a_0, a_1, \ldots\}$ and a finite set $B = \{b_0, b_1, \ldots, b_{M-1}\}$ of size M, the recursive indexing of A by B is a mapping I of A to the collection of all sequences of symbols from B such that

$$I(a_i) = \underbrace{b_{M-1} b_{M-1} \ldots b_{M-1}}_{q \text{ times}} b_r \text{ if } i = q(M-1) + r \tag{1}$$

where q and r are the quotient and remainder of i when divided by $M-1$. Set B is called the representation set. Defined as such, recursive indexing is a one-to-one mapping, a symbol-to-variable length, M-ary, prefix-free code and therefore uniquely and instantaneously decodable.

Since the second stage of the recursively indexed binary encoder is an optimum symbol-to-variable length encoder (the Huffman algorithm is used to design such), the statistic of the representation symbols must be computed. For this purpose we first compute the number of representation symbols needed to describe a typical source sequence $X_1, X_2, \ldots X_n$ of the length n from set A. Define $p_k = Pr(X = a_k)$, then the number n_0 of the occurrences of symbol b_0 is computed as follows. Observe that b_0 occurs once whenever $a_0, a_{M-1}, a_{2M-2}, \ldots, a_{k(M-1)}, \ldots$ occur. The number of times these symbols occur is given by $np_0, np_{M-1}, np_{2M-2}, \ldots, np_{k(M-1)}, \ldots$ and so on. Therefore,

$$n_0 = n \sum_{k=0}^{\infty} p_{k(M-1)}. \tag{2}$$

In a similar manner the number n_j of the occurrences of symbol b_j are found to be:

$$\begin{aligned} n_j &= n \sum_{k=0}^{\infty} p_{k(M-1)+j} \text{ for } j = 0, 1, \ldots, M-2. \\ n_{M-1} &= n \sum_{k=0}^{\infty} \sum_{j=0}^{M-1} k p_{k(M-1)+j}. \end{aligned} \tag{3}$$

From these it is seen that on the average the number of representation symbols needed for n source symbols is

$$\sum_{j=0}^{M-1} n_j \tag{4}$$

Therefore, the average number of representation symbols to represent one source symbol is

$$\frac{1}{n} \sum_{j=0}^{M-1} n_j = 1 + \frac{n_{M-1}}{n} = 1 + \sum_{k=0}^{\infty} \sum_{j=0}^{M-1} k P_{k(M-1)+j} \tag{5}$$

It is convenient to define the above expression as the expansion factor, denoted e, of the recursive indexing I. It is the factor by which one source symbol is expanded by the recursive indexing. The relative frequency q_j of representation symbol b_j, is computed as follows:

$$q_j = \frac{n_j}{n}. \tag{6}$$

An optimum symbol-to-variable length binary encoder after the recursive indexing takes one representation symbol at a time and produces the corresponding binary sequence from a set of variable length code words. It is designed for example using the Huffman algorithm. Then its rate R_{RI} , the number of binary digits per representation symbol, is bounded by

$$H(B) \leq R_{RI} < H(B) + 1 \tag{7}$$

symbol-to-variable length code by the Huffman algorithm it is observed that rate R is almost equal to $H(B)$, the lower bound, when M is large.

The overall rate of the recursively indexed binary encoder then is bounded

$$eH(B) \leq R < eH(B) + e \tag{8}$$

Note that e approximately equals 1 if M is large.

3 DPCM with Recursively Indexed Binary Encoding

3.1 Source and DPCM

Let us consider the encoding by a DPCM system of a first-order Gauss-Markov process,

$$X_k = \rho X_{k-1} + W_k \tag{9}$$

where W_k is an independent identically distributed Gaussian with mean zero and variance σ_w^2 . The source correlation coefficient ρ is between -1 and 1.

The DPCM system consists of a quantizer, a predictor and a binary encoder In a typical operational cycle the difference Z_k between the source output X_k and its prediction $\hat{X}_k$ is quantized by quantizer Q yielding $Q(Z_k)$, which in turn is binary encoded. The predictor considered in this paper is a first-order linear predictor and therefore it is given by bY_{k-1} for some constant b.

We will say that DPCM is matched to the source if the source correlation ρ equals the predictor coefficient b and that it is mismatched otherwise.

The binary encoder in ordinary DPCM is either a fixed-to-fixed length or a fixed-to-variable length binary encoder. In the former the binary encoder takes the index of the quantizer output level, produces its binary representation and sends the binary sequence through the channel. In the latter blocks of quantizer outputs are buffered and (usually) entropy-encoded.

The performance of a DPCM system will be measured by distortion and rate. The distortion incurred is defined to be

$$D = \limsup_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} E\Big\{(X_k - Y_k)^2\Big\}. \tag{10}$$

It is well-known that the error of the DPCM system is that incurred by the quantizer alone and nowhere else, i.e., $X_k - Y_k = Z_k - Q(Z_k)$ Hence the distortion can be rewritten as

$$D = \limsup_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} E\Big\{(Z_k - Q(Z_k))^2\Big\}. \tag{11}$$

The rate is defined to be the average number of binary digits used to transmit one source symbol. In case of a fixed-to-fixed length binary encoder is given by $\lceil \log_2 N \rceil$, where N is the number or quantizer output levels. In case of a fixed-to-variable length entropy encoder it is approximately H(Q), the entropy of the quantizer output process.

3.2 Recursively Indexed DPCM

The DPCM system considered in this paper is a recursively indexed DPCM system. It is different from an ordinary system in the following two ways:

1. The quantizer Q is an infinite level uniform quantizer with the thresholds being the mid-points of output levels.
2. The binary encoder is a recursively indexed fixed-to-variable length encoder.

The quantizer with infinitely-many output levels uniformly spaced yields granular distortion only. The magnitude of the distortion is bounded by $\Delta/2$. Therefore, no matter how large the input to the quantizer is, due to bad prediction, its output is at most $\Delta/2$ different from its input. Since an unoverloaded quantized value is available to the predictor at the next prediction, the system can track the source output, thereby yielding lower prediction errors. Due to this quick response the system does not have catastrophic error propagation, which DPCM with a finite number of quantizer output levels has when a pathological source sequence is encountered.

As discussed in Section 2, a recursively indexed binary encoder is necessary because the quantizer used has infinitely many output levels. We note that the encoder is not necessarily an entropy-encoder for the quantized process.

The distortion for this system is given by (11), while the rate is simply the entropy of the representation symbols multiplied by the expansion factor.

4 Simulation Results

To test this system first order Gauss-Markov and Laplace-Markov random number generators with correlation coefficient ρ were used. The Laplace-Markov process was defined as in [6]. The variance of the innovation sequence was chosen so as to get a source variance of unity. For each realization of the process 100,000 samples were used.

4.1 Rate-Distortion Performance

To obtain the rate-distortion performance, the predictor was matched to the source correlation coefficient and the step-size was varied. Each step-size Δ generated a distortion-rate pair which was then plotted. The results were overlaid on the optimum results from [6]. The results for $\rho = 0.8$ are plotted in Figures 1 and 2.

For the Gauss-Markov source the rate distortion performance of the proposed system achieves (or almost achieves) the optimum performance for the entire range of rates. This is true for both the experimentally obtained rates from [6] as well as the asymptotic results. Recall that for the optimum results the quantizer was designed using a relatively complex iterative procedure, while for the proposed system the quantizer was designed by simply specifying the step-size.

For the Laplace-Markov source, the proposed scheme again performs as well as the optimum scheme for almost all rates as far as the experimentally obtained results are concerned. However, the proposed scheme provides *better* results than the asymptotic results in [6]. We presume this is due to an error in the asymptotic results (or our interpretation of them!).

Similar results were obtained for ρ= 0.5, and 0.2 for both sources. The results seem to indicate that the problem of matching the quantizer with the input statistics can be easily resolved by the use of a recursively indexed quantizer. The quantizer can be easily designed by simply specifying the step-size Δ, which in turn can be specified based on the distortion requirements.

4.2 Performance Under Mismatch Conditions

To investigate the effect of mismatch between the predictor and model coefficients we used three values for the spacing Δ of the uniform quantizer: $2.5\sigma_x, 1.5\sigma_x, 0.2\sigma_x$. These correspond to low, medium, and high resolution (rate) quantizations, respectively. For each spacing for uniform- quantization, the source sequence is applied to the recursively indexed DPCM system with various values for the predictor coefficient. The quantizer output sequence is fed to the recursively indexed binary encoder with M representation symbols. The value of M ranges from 5 to 31. The results are shown in Figure 3, where the horizontal and vertical axis are respectively the predictor coefficient and the product of the rate and distortion.

For the Gauss-Markov source, for low rate ($\Delta = 2.5\sigma_x$) the distortion is about 0.44. The best performance is obtained when the predictor coefficient is around 0.6, where the rate is 0.58 bits/sample. This value is far below the source correlation 0.8. Note that in [4], the best performance was reported around 0.815, slightly higher than the source correlation. If 5% increase in rate is allowed, then Figure 3 shows that the predictor coefficient can be anywhere between 0.4 and 0.8.

For medium rate ($\Delta = 1.5\sigma_x$) the distortion is about 0.186, while $\Delta^2/12$ is 0.1875. The best predictor coefficient is around 0.7. The rate for this value of Δ and b is 1.12 bits/sample. The value of b is again lower than the source correlation. Again a 5% increase in rate allows the predictor coefficient anywhere between 0.48 and 0.88. For high rate ($\Delta = 0.2\sigma_x$) the distortion is about 0.00333, while $\Delta^2/12$ is 0.00333. The best predictor coefficient is around 0.8, where the rate is 4.2 bits/sample, and can range from 0.0 to 0.99 if 5% increase in rate is allowed.

We observe that the best predictor coefficient moves from below closer to the source correlation as resolution increases. This may be because for larger values of Δ the quantization noise tends to be magnified when the predictor coefficient is larger[1]. Also we note that the distortion expression ($\Delta^2/12$) is quite accurate even for large Δ. Similar results are observed for the Laplace-Markov source.

4.3 Effect of the Alphabet Size M

Finally we look at the size of the representation alphabet on the rate. Note that the larger the representation alphabet is for a certain value of Δ the less likely it is that the encoder will enter the recursive mode. This implies that given a value of Δ larger values of M will tend to lower the value of the expansion factor e closer to 1, thus lowering the rate. If this is a very strong effect, recursive indexing looses some of its charm, as the smaller alphabet size of the reproduction alphabet makes it more amenable to entropy coding. The recursively indexed DPCM system was simulated with representation alphabet sizes 7,9,11,13, and 15. The results for a Laplace-Markov source are shown in Figure 4. Note that at low rates (large values of Δ)there is no difference between these sizes. At higher rates, there is noticeable difference as we increase the alphabet size from 7 to 11. After that point there is very little improvement obtained in the rate when the alphabet size is increased.

5 Conclusions

Recursively indexed DPCM, which features an infinite level quantizer coded with a finite alphabet entropy coder has been shown to be an efficient encoder for first order Gauss-Markov and Laplace-Markov sources. The use of a recursively indexed quantizer in a standard DPCM system seems to provide a solution both to the problem of matching the quantizer to the prediction error statistics, and the problem of exactly matching the predictor to the source, at least for these simple sources.

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